Selected scientific achievements of Professor Zygmunt Zahorski*

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"I like only difficult topics, the easy ones I leave for the beginners (who has done as well sometime something difficult anyway). In any case, to deal with problems more difficult than the ones that I have done. It is yet a pleasure (of struggle with difficulties) joined with an unpleasure when you are completely helpless."

Zygmunt Zahorski

Scientific creativity of Professor Zygmunt Zahorski was concentrated around the real analysis and trigonometric series. Below we present only three selected topics, in our opinion the most prestigious from among Professor Zahorski's achievements.

The most famous publication of Professor Zahorski, entitled *Sur la premiére dérivée, Trans. Amer. Math. Soc.* ***69****, no. 1 (1950), 1–54*, which brought him the well-deserved fame and great esteem, belongs to the specially selected set of works of Polish mathematicians with the biggest number of citations (and what is more, Professor Zahorski is, after Stefan Banach, one of the most frequently cited Polish mathematician – according to the research carried out by Polish Mathematical Society). This paper made, without any doubt, a quantum jump in the direction of recognizing the properties of derivatives. In the context of a problem of characterization of the Lebesque sets for derivatives of continuous functions (that is the sets of form $\{x\colon f'(x) > a\}$ and $\{x\colon f'(x) < a\}$) Professor Zahorski identified in this paper the famous descendent sequence of classes of the sets M_k, $k = 0, 1, \ldots, 5$ and the respective descendent sequence of classes of functions $\mathcal{M}_k$, $k = 0, 1, \ldots, 5$. These classes are routinely called as the Zahorski classes. A well-known American expert in the field of real functions theory – Andrew Bruckner – has devoted to these classes one of the

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chapters in his famous monograph *Differentiation of Real Functions, Springer, 1978.* It is worth to mention that the Zahorski classes M_k and $\mathcal{M}_k$ are not only included in the archives of great mathematical achievements, but they are still the subject of discussions and investigations (we have been personally convinced about this as the participants of International Summer Conference on Real Functions Theory organized for many years by the Slovak Academy of Sciences, University of Łódź, Łódź University of Technology and Pomeranian University in Słupsk).

The next important paper, considering Professor's achievements, is the paper: *Une série de Fourier permutée d'une fonction de classe L^2 divergente presque partout, C. R. Acad. Sci. Paris* ***251*** *(1960), 501–503.*

Professor S. Lipiński wrote as follows:

> It contained the proof of Kołmogorov theorem: *There exists a square integrable function, the (trigonometric – editorial note) Fourier series of which, after some rearrangement of terms, is divergent almost everywhere (in other words, the trigonometric orthogonal system is not absolutely convergent almost everywhere – editorial note)*, published without proof in paper *A. Kolmogoroff, D. Menschoff, Sur la convergence des series de fonctions orthogonales, Math. Zeitschr.* ***26*** *(1927), 432–441.* Despite numerous queries and requests for some advices, how to carry out the proof, Kołmogorov did not publish the proof and all the hints, given by him, were never enough to conduct the proof. Neither the function, nor the permutation of series were known. Finally Zahorski found them and consequences of this discovery are significant. Immediately after publication, Zahorski's paper served as an exemplar to P. Uljanow: *P.L. Uljanov, Raschodjaščiesja rjady Furie, Usp. Mat. Nauk* ***16*** *(1961), 61–142* and *P.L. Uljanov, Raschodjaščiesja rjady po sisteme Haara i po bazisam, DAN SSSR* ***138*** *(1961), 556-559*, who used Zahorki's method for the Haar and Walsh orthogonal series, and next for any orthogonal systems complete in L^2.[1] Afterwards, A.M. Olewski in his paper *Raschodjaščiesja rjady iz L^2 po polnym sistemam, DAN SSSR 138 (1961), 545–548*, did the same for the expansions with respect to any base in L^2.

The following paper by L. Tajkov deserves also for attention: *L.W. Tajkow, O raschodimosti rjadow Furie po pieriestawlennoj trigonometriczeskoj sistemie, Usp. Mat. Nauk* ***18*** *(1963), 191–198.* It has been proven there (with some surplus but also extraordinarily elegantly) the following theorem: there exist a couple of conjugated functions $F(x)$ and $\tilde{F}(x)$ belonging to $\bigcap_{p\in\mathbb{N}} L_p[0,2\pi]$, the Fourier series of which, after some rearrangement (the same for both series) are divergent to $\pm\infty$ for every $x \in [0, 2\pi]$. Discussion and summary of the results (obtained more or less till 1982) concerning "divergence of the Fourier series" with respect to various orthogonal systems, by including and emphasizing the importance of Professor Zahorski's paper, can be found in paper *P.L. Uljanow, A.N. Kołmogorow i raschodjaščiesja rjady Furie, Usp. Mat. Nauk* ***38*** *(1983), 51–90*[2]. Moreover, in the context of this paper it is worth to notice the, not highlighted there, result of Adriano M. Garsia (remarkable American specialist in combinatorics) which is, in some sense, dual to Professor Zahorski's result. Thus, in paper *Adriano M. Garsia, Existence of almost everywhere convergent rearrangements for Fourier series of L_2 functions, Annals Math.* ***79****, no. 3*

[1] Many valuable pieces of information devoted to tis subject can be found in monograph *B.I. Gołubow, A.W. Efimow, W.A. Skworcow, Walsh series and transformations. Theory and applications, Nauka, Moscow 1987.*

[2] Valuable reference is also the monograph by B.S. Kashin and A.A. Saakyan: *Orthogonal series, Nauka, Moskow 1984*, reviewed by the way by P.L. Uljanow. However the authors of this monograph were concentrated rather on just the contents of presented lecture and less on its connections with the source literature.

(1964), 623–629 he proved that the Fourier series of every function of class $L_2[-\pi,\pi]$ can be rearranged such that the obtained new series can be convergent almost everywhere in interval $[-\pi,\pi]$. And, what is more, Garsia proved (what is in fact the main result of cited paper) that the permutations on $\mathbb{N}$, rearranging the given Fourier series to the series convergent almost everywhere, are, among others, the "almost all" permutations p on the set of natural numbers satisfying condition

$$p\left([m_k, m_{k+1})\right) = [m_k, m_{k+1})$$

for $k = 1, 2, \ldots$, where $m_1 := 1$ and $\{m_k\}_{k=1}^{\infty}$ is the "sufficiently fast" increasing sequence of natural numbers. Since we consider here a family of the simplest permutations on $\mathbb{N}$ (representing the composition of permutations on the successive intervals of $\mathbb{N}$), thus the Garsia result turned on once more the green light on the road leading to the solution of the, presented below, Luzin hypothesis.

Remark. Sequence $\{m_k\}_{k=1}^{\infty}$, indicated above, can be dependent on specific function f, since in the proof of Garsia's result it is only required that

$$S_{m_k}(x, f) \to f(x) \quad \text{(almost everywhere)}$$

(traditionally, symbol $S_n(x, f)$ denotes here the n-th partial sum of the Fourier series of function $f(x)$). However it is known that (see volume II, pages 164 and 165 in Antoni Zygmund's monograph *Trigonometric series, Cambridge University Press, Cambridge 2002*) if $\{n_k\}_{k=1}^{\infty}$ is the increasing sequence of natural numbers such that $n_{k+1}/n_k > q > 1$ for $k = 1, 2, \ldots$, then for every function $f \in L_2[-\pi,\pi]$ we have

$$S_{n_k}(x, f) \to f(x) \quad \text{(almost everywhere)}.$$

For example, we have $S_{2^k}(x, f) \to f(x)$ (almost everywhere) for every function $f \in L_2[-\pi,\pi]$.

A. Zygmund payed also his attention to the fact that if $f \in L_2[a, b]$, then $S_n(x, f) - f(x) \to 0$ in $L_2[a, b]$, which yet implies the existence of some increasing sequence $\{m_k\}_{k=1}^{\infty}$ of natural numbers such that

$$S_{m_k}(x, f) \to f(x) \quad \text{(almost everywhere)}.$$

Proof of this fact is also included in monograph *W. Rudin, Real and Complex Analysis, PWN, Warsaw 1998 (in Polish)*, (see Theorem 3.12, the proof of which is carried out within the framework of proof of Theorem 3.11).

The most serious problem, which Professor Zahorski confronted, was the attempt of proving the Luzin hypothesis from 1913, saying that the Fourier series of square integrable function is convergent almost everywhere. Many prominent mathematicians made the efforts of answering the question whether this hypothesis is true or not, however without success. Professor Zahorski failed as well in his "trials", although he was working on this problem, with breaks, almost till the end of his life. In a rush – like he admitted himself – he published, also incorrect, proof of the Luzin hypothesis correctness, which for sure affected adversely his research. The Luzin hypothesis has been proven by Swedish mathematician Lennart Carleson in his, yet historical today, paper: *L. Carleson, On convergence and growth of partial sums of Fourier series, Acta Math.* ***116*** *(1966), 1–48*. In 1992 Carleson was awarded for this result, among others, with the Wolf Prize (reprint of this paper, as well as the deep discussion of other Carleson's scientific achievements can be found in excellent book edited by S.S. Chern and F. Hirzebruch: *Wolf Prize in Mathematics, vol. 1, 91–142, World Scientific, Singapore 2000*). The "obstacle" to being awarded with the Fields Medal for this first-class result was Carleson's age. Carleson exceeded for 2 years the age limit – 36 years – imposed on the Fields Medal winners. It is worth to mention now, for the contrast with Carleson's result, the Kolmogorov theorem: there exists function $f \in L(0, 2\pi)$, the Fourier series of which is divergent almost everywhere (connection between Carleson and Kolmogorov results will recur at the end of this paper, in the final remark). Proof of the Kolmogorov theorem can be found, among others, in the first volume of mentioned above A. Zygmund's monograph, on pages 310–314 (see also the cited above P.L. Uljanow paper from 1983 devoted anyway to the "connections" of A. Kolmogorov with the divergent Fourier series). Next, D.E. Menszov in the 30s proved the following theorem: every 2π-periodic, measurable and finite function $f(x)$ can be expanded into the trigonometric series which is almost everywhere convergent in $[0, 2\pi]$. About this one, as well as about some other Menszov's results in this subject, one can read in paper: *A.N. Kołmogorow, S.M. Nikolski, W.A. Skworcow, P.L. Uljanow, Dmitrij Jewgeniewicz Mienszow (paper on the occasion of 90th birthday anniversary), Usp. Mat. Nauk* ***37*** *(1982), 209–219.*

Many facts and pieces of information presented in this paper have been elaborated, among others, on the basis of the following sources, in which some other Professor Zahorski's results are also discussed:

- J.S. Lipiński, *Zygmunt Zahorski's papers on the real functions theory, Zesz. Nauk. PŚ., Mat.-Fiz.,* ***48*** *(1986), 29–38 (in Polish, Scientific Notes of Silesian University of Technology, series Mathematics-Physics)*. Let us add that this scientific note was the special issue of this journal dedicated to Professor Zahorski on the occasion of his 70th birthday anniversary and many famous mathematicians published there their works. For example, Professor Józef Siciak published the paper (in Polish), in which the author generalized for the case of functions $f\colon \mathbb{R}^n \to \mathbb{R}$ of class C^∞ the set-theoretic characterization of the sets of singular points of these functions, given for case $n = 1$ by Professor Zahorski. Jean Schmets and Manuel Valdivia in their paper *The Zahorski theorem is valid in Gevrey classes, Fund. Math.* ***151*** *(1996), 149–166* obtained the generalization of Siciak's result for the functions of Gevrey class.
- note entitled: *Zygmunt Zahorski – an obituary, Real Anal. Exchange* ***23****, no. 2 (1999), 359–362 (available online).*
- W. Wilczyński, *Zygmunt Zahorski and contemporary real analysis, paper in the current monograph: R. Wituła, D. Słota, W. Hołubowski (eds.), Monograph on the occasion of* 100^{th} *birthday anniversary of Zygmunt Zahorski, Wyd. Pol. Śl., Gliwice 2015.*

Final remarks

1. At the end it is worth to mention that the Carleson result, concerning the almost everywhere convergence of the trigonometric Fourier series of the square integrable function on (for definiteness) one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, possesses some significant generalizations:

   - first, R.A. Hunt in his paper *On the convergence of Fourier series, 1968 Orthogonal Expansions and their Continuous Analogues (Proc. Conf. Edwardsville, 1967), 235–255, Southern Illinois Univ. Press, Carbondale*, generalized Carleson's result for the Fourier series of functions of class L^p, $1 < p < \infty$;
   - next, N.Y. Antonov in his paper *Convergence of Fourier series, East J. Approx.* ***2*** *(1996), 187–196*, proved that the Fourier series of every function from the Lorentz space $L \log L \log\log\log L(\mathbb{T})$ is convergent almost everywhere;
   - some generalization of Antonov's result has been obtained, among others, in paper *M.J. Carro, M. Mastyło, L. Rodriguez-Piazza, Almost everywhere convergent Fourier series, J. Fourier Anal. Appl.* ***18*** *(2012), 266–286.*

2. A.M. Garsia in his, cited above, paper (1964, Annals Math.) showed the following general result: if $\{f_k\}$ is an orthonormal system of scalar functions in some Hilbert space and $\{a_k\} \in l_2$, then there exists a permutation $p : \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{k=1}^{\infty} a_{p(k)} f_{p(k)}$ converges almost surely (for shortening we will use hereafter the convenient notation "a.s.").
   Next, E.M. Nikishin in his paper *The convergence of rearrangements of series of functions, Math. Zametki* ***1****, no. 2 (1967), 129–136, English translation in Math. Notes* ***1****, no. 2 (1967), 85–90*, proved the following generalization of Garsia's theorem: if a series $\sum_{k=1}^{\infty} \xi_k$ of scalar random variables converges in measure to a random variable S and $\sum_{k=1}^{\infty} |\xi_k|^2 < \infty$ a.s., then there exists a rearrangement $\sum_{k=1}^{\infty} \xi_{p(k)}$ of this series convergent a.s. to S.
   Finally S. Levental, V. Mandrekar and S.A. Chobanyan in paper *Towards Nikishin's Theorem on the Almost Sure Convergence of Rearrangements of Functional Series, Funct. Anal. Appl.* ***45****, no. 1 (2011), 33-45* proved the following theorem.

   Theorem. Let $\{k_n\}_{n=1}^{\infty}$ be an increasing sequence of positive integers. If the sequence of partial sums $S_{k_n} = \sum_{l=1}^{k_n} \xi_l$ of a series $\sum_{l=1}^{\infty} \xi_l$ of random variables (taking the values from a normed space, in general) converges a.s. to a random variable S, then there exists a rearrangement $\sum_{l=1}^{\infty} \xi_{p(l)}$ of the series convergent a.s. to S, provided that

   $$\sum_{l=k_n}^{k_{n+1}} \xi_l r_l \to 0 \quad \text{a.s.} \quad \text{for} \quad n \to \infty,$$

   where $\{r_l\}_{l=1}^{\infty}$ is a sequence of Rademacher random variables independent of $\{\xi_l\}_{l=1}^{\infty}$.

Moreover, they showed that under this condition (and only under this condition) the set of $\{k_n\}-$simple permutations $p : \mathbb{N} \to \mathbb{N}$ (which means that $k_1 := 1$, $p([k_n, k_{n+1})) = [k_n, k_{n+1})$, $n \in \mathbb{N}$), for which the series $\sum_{l=1}^{\infty} \xi_{p(l)}$ converges a.s. to S has the full measure (the respective measure is defined in the final part of Professor W. Wilczyński's paper included in this monograph).
It should be also emphasized that a random variable taking the values from a normed space is understood in the following way. Let $(\Omega, \mathfrak{M}, p)$ denote the underlying probability space. Let $(X, ||\cdot||)$ be a Banach space over $\mathbb{R}$ or $\mathbb{C}$. A mapping $\xi : \Omega \to X$ is called a random variable taking the values from X if it is Bochner measurable, i.e. $\xi = \lim \xi_n$ $p--$a.s. where $\xi_n : \Omega \to X$ is measurable, and takes only finitely many values for every $n \in \mathbb{N}$.

3. Let

$$D_f := \{x \in [0, 2\pi] : \ \{S_n(x; f)\} \text{ diverges}\}$$

for $f \in C(\mathbb{T})$. It is easy to prove that D_f is a $G_{\delta\sigma}$ set. By Carleson's theorem D_f has measure zero. Professor J. Śladkowska-Zahorska in paper *Sur l'ensemble des points de divergence des series de Fourier des fonctions continues, Fund. Math.* ***49*** *(1961), 271–294*, proved the following partial converse.

Theorem. Let $B \subseteq (0, 2\pi)$ be an F_σ set of logarithmic measure zero (i.e. for each $\varepsilon > 0$ there is a sequence $\{I_n\}$ of intervals with $B \subset \bigcup I_n$, $|I_n| < L_n < 1$ and $\sum 1/|\log L_n| < \varepsilon$) and let $A \subseteq B$ be a $G_{\delta\sigma}$ set. Then there exists $f \in C(\mathbb{T})$ with $D_f = A$.

M. Ajtai and A.S. Kechris in their paper *The set of continuous functions with the everywhere convergent Fourier series, Trans. Amer. Math. Soc.* ***302****, no. 1 (1987), 207–221*, used this fact to give the proof of the following theorem.

Theorem. The set $EC \subset C(\mathbb{T})$ of continuous functions with everywhere convergent Fourier series is a complete coanalytic set. In particular, it is not a Borel set in $C(\mathbb{T})$.